Generalized Equivalence Principle in Extended New General Relativity

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In extended new general relativity, which is formulated as a reduction of $\overline{\text{Poincaré}}$ gauge theory of gravity whose gauge group is the covering group of the Poincaré group, we study the problem of whether the total energy-momentum, total angular momentum and total charge are equal to the corresponding quantities of the gravitational source. We examine this for charged axi-symmetric solutions of gravitational field equations. Our main concern is the restriction on the asymptotic form of the gravitational field variables imposed by the requirement that physical quantities of the total system are equivalent to the corresponding quantities of the charged rotating source body. This requirement can be regarded as an equivalence principle in a generalized sense.

§1. Introduction

The energy-momentum, angular momentum and electric charge play central roles in modern theoretical physics. The conservations of the first two are related to the homogeneity and isotropy of space-time, respectively, and the charge conservation corresponds to the invariance of the action integral under internal $U(1)$ transformations. Also, local objects such as energy-momentum density, angular momentum density and charge density are well defined if gravitational fields do not take part in.

In general relativity (G.R.), however, well-behaved energy-momentum and angular momentum densities have not been defined, but total energy-momentum and total angular momentum are defined for asymptotically flat space-time surrounding an isolated finite system. The total energy of the system is regarded as the inertial mass multiplied by the square of the velocity of light, and there arises the question: Is the active gravitational mass of the isolated system equal to its inertial mass? This is an aspect of the equivalence principle, and it is usually said to be affirmatively settled in G.R.[1, 2] But, the *equality* of the active gravitational mass and the inertial mass is violated for the Schwarzschild metric when it is expressed with the use of a certain coordinate system.[3]

New general relativity (N.G.R.),[4] which has been formulated by gauging coordinate translation and is underlain with Weitzenböck space-time, is a possible alternative to G.R. The most general gravitational Lagrangian density, which is quadratic in torsion tensor and is invariant under global Lorentz transformations including also space inversion and general coordinate transformations, has three free parameters c_1 , c_2 and c_3. Solar system experiments show that c_1 and $-c_2$ are very likely to be equal to $-1/(3\kappa)$: $c_1 = -1/(3\kappa) = -c_2$. In Ref. 5, Shirafuji, Nashed and Hayashi have found, for the case with $c_1 = -1/(3\kappa) = -c_2$, the most general spherically symmetric solution, and they clarified the restriction on the behavior of the vierbeins at spatial infinity imposed by the requirement that the inertial mass is equal to the active gravitational mass. The consideration has been extended to the case with $c_1 \neq -1/(3\kappa) \neq -c_2$ in Ref. 2.

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Extended new general relativity (E.N.G.R.),[6] is obtained as a reduction of $\overline{\text{Poincaré}}$ gauge theory ($\bar{\text{P}}$.G.T.) of gravity[7] which is formulated on the basis of the principal fiber bundle over the space-time, possessing the covering group $\bar{P}_0$ of the Poincaré group as the structure group, by following the lines of the standard geometric formulation of Yang-Mills theories as closely as possible. E.N.G.R.[6] is also underlain with Weitzenböck space-time and has many in common with N.G.R. The field equations for the vierbeins in E.N.G.R. , for example, are identical with those in N.G.R., if fields with non-vanishing "intrinsic" energy-momentum do not take part in.

In this paper, considering charged axi-symmetric solutions in E.N.G.R., we examine the condition imposed on the asymptotic behaviors of the field variables by the requirement[*] that the total energy-momentum, the total angular momentum and the total charge of the system are all equal to corresponding quantities of the central gravitating body.

We organize this paper as follows. In §2, we give a basic formulation of E.N.G.R. In §3, we calculate conserved quantities for a charged axi-symmetric solution. In §4, we study solutions obtainable from the solution discussed in §3 and examine restrictions on the field variables by the requirement mentioned above. In the final section, we give a summary and discussions.

§2. Basic formulation

2.1. *$\overline{\text{Poincaré}}$ gauge theory*

$\bar{\text{P}}$.G.T.[7] is formulated on the basis of the principal fiber bundle $\mathcal{P}$ over the space-time M possessing the covering group $\bar{P}_0$ of the proper orthochronous Poincaré group as the structure group. The space-time is assumed to be a noncompact four dimensional differentiable manifold having a countable base. The bundle $\mathcal{P}$ admits a connection Γ, whose translational and rotational parts of the coefficients will be written as $A^k{}_\mu$ and $A^k{}_{l\mu}$, respectively. The fundamental field variables are $A^k{}_\mu$, $A^k{}_{l\mu}$, the Higgs-type field $\psi = \{\psi^k\}$ and matter field $\phi = \{\phi^A | A = 1, 2, 3, \cdots, N\}$.[**] These fields transform according as[***]

$$
\begin{aligned}
\psi'^k &= (\Lambda(a^{-1}))^k{}_l(\psi^l - t^l) \ , \\
A'^k{}_\mu &= (\Lambda(a^{-1}))^k{}_l(A^l{}_\mu + t^l{}_{,\mu} + A^l{}_{m\mu}t^m) \ , \\
A'^k{}_{l\mu} &= (\Lambda(a^{-1}))^k{}_m A^m{}_{n\mu}(\Lambda(a))^n{}_l + (\Lambda(a^{-1}))^k{}_m(\Lambda(a))^m{}_{l,\mu} \ , \\
\phi'^A &= (\rho((t,a)^{-1}))^A{}_B\phi^B \ ,
\end{aligned}
\tag{2·1}
$$

under the $\overline{\text{Poincaré}}$ gauge transformation

$$
\sigma'(x) = \sigma(x) \cdot (t(x), a(x)) \ , \quad t(x) \in T^4 \ , \quad a(x) \in SL(2, C) \ . \tag{2·2}
$$

[*] This is a generalization of the requirement that the inertial mass is equal to the active gravitational mass.

[**] Unless otherwise stated, we use the following conventions for indices: The middle part of the Greek alphabet, $\mu, \nu, \lambda, \cdots$, refers to 0, 1, 2 and 3, while the initial part, $\alpha, \beta, \gamma, \cdots$, denotes 1, 2 and 3. In a similar way, the middle part of the Latin alphabet, $i, j, k, \cdots$, means 0, 1, 2 and 3, unless otherwise stated. While the initial part, $a, b, c, \cdots$, denotes 1, 2 and 3. The capital letters A and B is used for indices for components of field ϕ, and N denotes the dimension of the representation ρ.

[***] For function f on M, we define $f_{,\mu} \overset{\text{def}}{=} \partial f / \partial x^\mu$.

Here, Λ is the covering map from $SL(2,C)$ to the proper orthochronous Lorentz group, and ρ stands for the representation of the $\overline{\text{Poincaré}}$ group to which the field ϕ is belonging. Also, σ and σ' stand for local cross sections of $\mathcal{P}$. Dual components $b^k{}_\mu$ of vierbeins $b^\mu{}_k \partial/\partial x^\mu$ are related to the field ψ and the gauge potentials $A^k{}_\mu$ and $A^k{}_{l\mu}$ through the relation

$$b^k{}_\mu = \psi^k{}_{,\mu} + A^k{}_{l\mu}\psi^l + A^k{}_\mu \ , \tag{2·3}$$

and these transform according as

$$b'^k{}_\mu = (\Lambda(a^{-1}))^k{}_l b^l{}_\mu \ , \tag{2·4}$$

under the transformation (2·2). Also, they are related to the metric $g_{\mu\nu}dx^\mu \otimes dx^\nu$ of M through the relation

$$g_{\mu\nu} = \eta_{kl} b^k{}_\mu b^l{}_\nu \tag{2·5}$$

with $(\eta_{kl}) \stackrel{\text{def}}{=} \text{diag}(-1,1,1,1)$.

There is a 2 to 1 bundle homomorphism F from $\mathcal{P}$ to affine frame bundle $\mathcal{A}(M)$ over M, and an extended spinor structure and a spinor structure exist associated with it.[8)] The space-time M is orientable, which follows from its assumed noncompactness and from the fact that M has a spinor structure.

The affine frame bundle $\mathcal{A}(M)$ admits a connection Γ_A. The T^4-part $\Gamma^\mu{}_\nu$ and $GL(4,R)$-part $\Gamma^\mu{}_{\nu\lambda}$ of its connection coefficients are related to $A^k{}_{l\mu}$ and $b^k{}_\mu$ through the relations

$$\Gamma^\mu{}_\nu = \delta^\mu{}_\nu \ , \quad A^k{}_{l\mu} = b^k{}_\nu b^\lambda{}_l \Gamma^\nu{}_{\lambda\mu} + b^k{}_\nu b^\nu{}_{l,\mu} \ , \tag{2·6}$$

by the requirement that F maps the connection Γ into Γ_A, and the space-time M is of the Riemann-Cartan type.

The field strengths $R^k{}_{l\mu\nu}$, $R^k_{\mu\nu}$ and $T^k{}_{\mu\nu}$ of $A^k{}_{l\mu}$, $A^k{}_\mu$ and of $b^k{}_\mu$ are given by *)

$$\begin{aligned} R^k{}_{l\mu\nu} &\stackrel{\text{def}}{=} 2(A^k{}_{l[\nu,\mu]} + A^k{}_{m[\mu}A^m{}_{l\nu]}) \ , \\ R^k{}_{\mu\nu} &\stackrel{\text{def}}{=} 2(A^k{}_{[\nu,\mu]} + A^k{}_{l[\mu}A^l{}_{\nu]}) \ , \\ T^k{}_{\mu\nu} &\stackrel{\text{def}}{=} 2(b^k{}_{[\nu,\mu]} + A^k{}_{l[\mu}b^l{}_{\nu]}) \ , \end{aligned} \tag{2·7}$$

and we have the relation

$$T^k{}_{\mu\nu} = R^k{}_{\mu\nu} + R^k{}_{l\mu\nu}\psi^l \ . \tag{2·8}$$

The field strengths $T^k{}_{\mu\nu}$ and $R^{kl}{}_{\mu\nu}$ are both invariant under *internal* translations. The torsion is given by

$$T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\nu\mu]} \ , \tag{2·9}$$

and the T^4- and $GL(4,R)$-parts of the curvature are given by

$$R^\lambda{}_{\mu\nu} = 2(\Gamma^\lambda{}_{[\nu,\mu]} + \Gamma^\lambda{}_{\rho[\mu}\Gamma^\rho{}_{\nu]}) \ , \tag{2·10}$$

$$R^\lambda{}_{\rho\mu\nu} = 2(\Gamma^\lambda{}_{\rho[\nu,\mu]} + \Gamma^\lambda{}_{\tau[\mu}\Gamma^\tau{}_{\rho\nu]}) \ , \tag{2·11}$$

*) We define

$$A_{\cdots[\mu\cdots\nu]\cdots} \stackrel{\text{def}}{=} \frac{1}{2}(A_{\cdots\mu\cdots\nu\cdots} - A_{\cdots\nu\cdots\mu\cdots}) \ ,$$

$$A_{\cdots(\mu\cdots\nu)\cdots} \stackrel{\text{def}}{=} \frac{1}{2}(A_{\cdots\mu\cdots\nu\cdots} + A_{\cdots\nu\cdots\mu\cdots}) \ .$$

respectively. Also, we have

$$T^k{}_{\mu\nu} = b^k{}_\lambda T^\lambda{}_{\mu\nu} = b^k{}_\lambda R^\lambda{}_{\mu\nu} \ , \tag{2·12}$$

$$R^k{}_{l\mu\nu} = b^k{}_\lambda b^\rho{}_l R^\lambda{}_{\rho\mu\nu} \ , \tag{2·13}$$

which follow from Eq.(2·6).

The covariant derivative of the matter field ϕ takes the form

$$\begin{aligned} D_k\phi^A &= b^\mu{}_k D_\mu \phi^A \ , \\ D_\mu\phi^A &\stackrel{\text{def}}{=} \partial_\mu\phi^A + \frac{i}{2}A^{lm}{}_\mu (M_{lm}\phi)^A + iA^l{}_\mu (P_l\phi)^A \ . \end{aligned} \tag{2·14}$$

Here, M_{kl} and P_k are representation matrices of the standard basis of the group $\bar{P}_0$: $M_{kl} = -i\rho_*(\overline{M}_{kl})$, $P_k = -i\rho_*(\overline{P}_k)$. The matrix P_k represents the intrinsic energy-momentum of the field ϕ^A,[8)] and it is vanishing for all the observed fields.[*)]

From the requirement of invariance of the action integral under internal $\bar{P}_0$ gauge transformations, it follows that the gravitational Lagrangian density is a function of T_{klm} and of R_{klmn}. The gravitational Lagrangian density agrees with that in Poincaré gauge theory, and hence *gravitational field equations take the same forms in these theories*. The field equation for the field ψ^k is automatically satisfied if those of $A^k{}_\mu$ and of ϕ^A are both satisfied, and ψ^k is a non-dynamical field in this sense.

2.2. *Extended new general relativity*

2.2.1. *Reduction of $\overline{\text{Poincaré}}$ gauge theory to an extended new general relativity*

In P̄.G.T., we consider the case in which the field strength $R^{kl}{}_{\mu\nu}$ vanishes identically,

$$R^{kl}{}_{\mu\nu} \equiv 0 \ , \tag{2·15}$$

and choose $SL(2.C)$-gauge such that

$$A^{kl}{}_\mu \equiv 0 \ . \tag{2·16}$$

Then, the curvature vanishes and the space-time reduces to Weitzenböck space-time which means that we have a teleparallel theory,

Also, vierbeins $b^k{}_\mu$, affine connection coefficients $\Gamma^\lambda{}_{\mu\nu}$ and torsion tensor $T^\lambda{}_{\mu\nu}$ are given by

$$b^k{}_\mu = \psi^k{}_{,\mu} + A^k{}_\mu \ , \tag{2·17}$$

$$\Gamma^\lambda{}_{\mu\nu} = b^\lambda{}_l \, b^l{}_{\mu,\nu} \ , \tag{2·18}$$

and

$$T^\lambda{}_{\mu\nu} = 2\,\Gamma^\lambda{}_{[\mu\nu]} = 2b^\lambda{}_k b^k{}_{[\mu,\nu]} = b^\lambda{}_k T^k{}_{\mu\nu} \ , \tag{2·19}$$

respectively. Introducing the volume element dv by

$$dv \stackrel{\text{def}}{=} \sqrt{-g}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{2·20}$$

[*)] In what follows, the field components $b^k{}_\mu$ and $b^\mu{}_k$ are used to convert Latin and Greek indices, similarly to the case of $D_k\phi^A$ and $D_\mu\phi^A$. Also, raising and lowering the indices $k, l, m, \cdots$ are accomplished with the aid of $(\eta^{kl}) \stackrel{\text{def}}{=} (\eta_{kl})^{-1}$ and (η_{kl}).

with $g \stackrel{\rm def}{=} \det(g_{\mu\nu}) = -\{\det(b^k{}_\mu)\}^2$, we consider the action integral

$$\mathbf{I} = \int_{\mathcal{D}} L(\psi^k{}_{,\mu}, \psi^k, A^k{}_{\mu,\nu}, A^k{}_\mu, A_{\mu,\nu}, A_\mu, \phi^A{}_{,\mu}, \phi^A, \phi^{*A}{}_{,\mu}, \phi^{*A})dv \ , \tag{2·21}$$

where A_μ and $\mathcal{D}$ denote electromagnetic vector potential and a compact region in M, respectively. *) Also, $*$ means the operation of complex conjugation, and thus ϕ^{*A} denotes the complex conjugate of ϕ^A.

We make the requirement:
(R.i) The action $\mathbf{I}$ is invariant under *local* internal translations and *global* $SL(2,C)$-transformations. From this, the identities[6]

$$\frac{\delta \mathbf{L}}{\delta \psi^k} + \partial_\mu \left(\frac{\delta \mathbf{L}}{\delta A^k{}_\mu} \right) + i \frac{\delta \mathbf{L}}{\delta \phi^A} (P_k \phi)^A - i \frac{\delta \mathbf{L}}{\delta \phi^{*A}} (P_k \phi)^{*A} \equiv 0 \ , \tag{2·22}$$

$$\mathbf{F}_k{}^{(\mu\nu)} \equiv 0 \ , \tag{2·23}$$

$${}^{\rm tot}\mathbf{T}_k{}^\mu - \partial_\nu \mathbf{F}_k{}^{\mu\nu} - \frac{\delta \mathbf{L}}{\delta A^k{}_\mu} \equiv 0 \ , \tag{2·24}$$

$$\partial_\mu {}^{\rm tot}\mathbf{S}_{kl}{}^\mu - 2 \frac{\delta \mathbf{L}}{\delta \psi^{[k}} \psi_{l]} - 2 \frac{\delta \mathbf{L}}{\delta A^{[k}{}_\mu} A_{l]\nu} - i \frac{\delta \mathbf{L}}{\delta \phi^A} (M_{kl}\phi)^A + i \frac{\delta \mathbf{L}}{\delta \phi^{*A}} (M_{kl}\phi)^{*A} \equiv 0 \ , \tag{2·25}$$

follow, where we have defined

$$\mathbf{L} \stackrel{\rm def}{=} \sqrt{-g} L \ , \tag{2·26}$$

$$\mathbf{F}_k{}^{\mu\nu} \stackrel{\rm def}{=} \frac{\partial \mathbf{L}}{\partial A^k{}_{\mu,\nu}} \ , \tag{2·27}$$

$${}^{\rm tot}\mathbf{T}_k{}^\mu \stackrel{\rm def}{=} \frac{\partial \mathbf{L}}{\partial \psi^k{}_{,\mu}} + i \frac{\partial \mathbf{L}}{\partial \phi^A{}_{,\mu}} (P_k \phi)^A - i \frac{\partial \mathbf{L}}{\partial \phi^{*A}{}_{,\mu}} (P_k \phi)^{*A} \ , \tag{2·28}$$

$$\begin{aligned} {}^{\rm tot}\mathbf{S}_{kl}{}^\mu \stackrel{\rm def}{=} & -2 \frac{\partial \mathbf{L}}{\partial \psi^{[k}{}_{,\mu}} \psi_{l]} - 2 \mathbf{F}_{[k}{}^{\nu\mu} A_{l]\nu} \\ & -i \frac{\partial \mathbf{L}}{\partial \phi^A{}_{,\mu}} (M_{kl}\phi)^A + i \frac{\partial \mathbf{L}}{\partial \phi^{*A}{}_{,\mu}} (M_{kl}\phi)^{*A} \ . \end{aligned} \tag{2·29}$$

By virtue of the identity (2·24), the density ${}^{\rm tot}\mathbf{T}_k{}^\mu$ can be expressed in the usual form of currrent in Yang-Mills theories:

$${}^{\rm tot}\mathbf{T}_k{}^\mu = \frac{\partial \mathbf{L}}{\partial A^k{}_\mu} \ . \tag{2·30}$$

When the field equations $\delta \mathbf{L}/\delta A^k{}_\mu \stackrel{\rm def}{=} \partial \mathbf{L}/\partial A^k{}_\mu - \partial_\nu (\partial \mathbf{L}/\partial A^k{}_{\mu,\nu}) = 0$ and $\delta \mathbf{L}/\delta \phi^A = 0$ are both satisfied, we have the following:
(i) The field equation $\delta \mathbf{L}/\delta \psi^k = 0$ is automatically satisfied, and hence ψ^k is not an

*) We consider the case in which electromagnetic field and charged field take part in. The field ϕ^A is considered to be a field with the electric charge q. This is a preparation to the subsequent sections in which space-time produced by a charged source is dealt with.

independent dynamical variable.
(ii)

$$\partial_\mu {}^{\text{tot}}\mathbf{T}_k{}^\mu = 0 \, , \tag{2·31}$$

$$\partial_\mu {}^{\text{tot}}\mathbf{S}_{kl}{}^\mu = 0 \, , \tag{2·32}$$

which are the differential conservation laws of the dynamical energy-momentum and of the "spin" angular momentum, respectively. These (i) and (ii) follow from Eqs.(2·22)∼(2·25).

Also, we require the following:
(R.ii) The Lagrangian density L is a scalar field on M.
Then, we have

$$\tilde{\mathbf{T}}_\mu{}^\nu - \partial_\lambda \mathbf{\Psi}_\mu{}^{\nu\lambda} - \frac{\delta \mathbf{L}}{\delta A^k{}_\nu} A^k{}_\mu - \frac{\delta \mathbf{L}}{\delta A_\nu} A_\mu \equiv 0 \tag{2·33}$$

and

$$\mathbf{\Psi}_\lambda{}^{(\mu\nu)} \equiv 0 \tag{2·34}$$

with

$$\tilde{\mathbf{T}}_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu{}^\nu \mathbf{L} - \mathbf{F}_k{}^{\lambda\nu} A^k{}_{\lambda,\mu} - \mathbf{F}^{\lambda\nu} A_{\lambda,\mu} - \frac{\partial \mathbf{L}}{\partial \phi^A{}_{,\nu}} \phi^A{}_{,\mu} - \frac{\partial \mathbf{L}}{\partial \phi^{*A}{}_{,\nu}} \phi^{*A}{}_{,\mu} - \frac{\partial \mathbf{L}}{\partial \psi^k{}_{,\nu}} \psi^k{}_{,\mu} \tag{2·35}$$

and

$$\mathbf{\Psi}_\lambda{}^{\mu\nu} \stackrel{\text{def}}{=} \mathbf{F}_k{}^{\mu\nu} A^k{}_\lambda + \mathbf{F}^{\mu\nu} A_\lambda \, , \tag{2·36}$$

$$\mathbf{F}^{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \mathbf{L}}{\partial A_{\mu,\nu}} \, . \tag{2·37}$$

The identities (2·33) and (2·34) lead to

$$\partial_\nu \tilde{\mathbf{T}}_\mu{}^\nu = 0 \, , \tag{2·38}$$

$$\partial_\nu \mathbf{M}_\lambda{}^{\mu\nu} = 0 \, , \tag{2·39}$$

when $\delta\mathbf{L}/\delta A^k{}_\mu = 0$, $\delta\mathbf{L}/\delta A_\mu = 0$, where $\mathbf{M}_\lambda{}^{\mu\nu} \stackrel{\text{def}}{=} 2(\mathbf{\Psi}_\lambda{}^{\mu\nu} - x^\mu \tilde{\mathbf{T}}_\lambda{}^\nu)$. Equations (2·38) and (2·39) are the differential conservation laws of the canonical energy-momentum and "extended orbital angular momentum",[9] respectively.

We also require the invariance of the action under the $U(1)$ gauge transformation

$$A'_\mu = A_\mu + \lambda_{,\mu} \, , \quad \phi'^A = \exp(iq\lambda)\phi^A \, , \quad \psi'^k = \psi^k \, , \quad A'^k{}_\mu = A^k{}_\mu \tag{2·40}$$

with λ being an arbitrary function on M, from which we can obtain

$$\mathbf{F}^{(\mu\nu)} \equiv 0 \, , \tag{2·41}$$

$$\mathbf{J}^\mu + iq \left(\frac{\partial \mathbf{L}}{\partial \phi^A{}_{,\mu}} \phi^A - \frac{\partial \mathbf{L}}{\partial \phi^{*A}{}_{,\mu}} \phi^{*A} \right) \equiv 0 \, , \tag{2·42}$$

$$\partial_\mu \left(\frac{\delta \mathbf{L}}{\delta A_\mu} - \mathbf{J}^\mu \right) \equiv 0 \tag{2·43}$$

with

$$\mathbf{J}^\mu \stackrel{\text{def}}{=} \frac{\partial \mathbf{L}}{\partial A_\mu} \, . \tag{2·44}$$

From the identity (2·43), the differential conservation law of electric charge,

$$\partial_\mu \mathbf{J}^\mu = 0 \ , \tag{2·45}$$

follows, when the field equation $\delta \mathbf{L}/\delta A_\mu = 0$ is satisfied. The functional dependence of L is restricted to be

$$L = \mathcal{L}(\psi^k, T_{klm}, F_{\mu\nu}, \nabla_k \phi^A, \phi^A, \nabla^*{}_k \phi^{*A}, \phi^{*A}) \tag{2·46}$$

with $\mathcal{L}$ satisfying certain identities,[6] where

$$F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{2·47}$$

$$\begin{aligned} \nabla_k \phi^A &\stackrel{\text{def}}{=} b^\mu{}_k \nabla_\mu \phi^A \stackrel{\text{def}}{=} b^\mu{}_k (\phi^A{}_{,\mu} + i A^k{}_\mu (P_k \phi)^A - iqA_\mu \phi^A) \ , \\ \nabla^*{}_k \phi^{*A} &\stackrel{\text{def}}{=} b^\mu{}_k \nabla^*{}_\mu \phi^{*A} \stackrel{\text{def}}{=} b^\mu{}_k (\phi^{*A}{}_{,\mu} - i A^k{}_\mu (P_k \phi)^{*A} + iqA_\mu \phi^{*A}) \ . \end{aligned} \tag{2·48}$$

The gravitational action *)

$$\tilde{\mathbf{I}}_G = \int_{\mathcal{D}} (c_1 \, t^{klm} t_{klm} + c_2 \, v^k v_k + c_3 \, a^k a_k) dv \tag{2·49}$$

with c_i $(i = 1, 2, 3)$ being real constants satisfies the requirements (R.i) and (R.ii). Here t_{klm}, v_k and a_k are irreducible components of the torsion tensor:

$$t_{klm} \stackrel{\text{def}}{=} \frac{1}{2}(T_{klm} + T_{lkm}) + \frac{1}{6}(\eta_{mk} v_l + \eta_{ml} v_k) - \frac{1}{3}\eta_{kl} v_m \ , \tag{2·50}$$

$$v_k \stackrel{\text{def}}{=} T^l{}_{lk} \ , \tag{2·51}$$

$$a_k \stackrel{\text{def}}{=} \frac{1}{6}\varepsilon_{klmn} T^{lmn} \ , \tag{2·52}$$

where ε_{klmn} stands for completely anti-symmetric Lorentz tensor with $\varepsilon_{(0)(1)(2)(3)} = -1$.**)

The action $\tilde{\mathbf{I}}_G$ with ***)

$$c_1 = -c_2 = -\frac{1}{3\kappa} \tag{2·53}$$

with κ being the Einstein gravitational constant is quite favorable experimentally.[4] In what follows, we shall assume that the condition (2·53) is satisfied. Thus, our gravitational action is

$$\mathbf{I}_G \stackrel{\text{def}}{=} \int_{\mathcal{D}} L_G dv \tag{2·54}$$

with

$$L_G \stackrel{\text{def}}{=} -\frac{1}{3\kappa}(t^{klm} t_{klm} - v^k v_k) + c_3 \, a^k a_k \ . \tag{2·55}$$

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*) This action agrees with the gravitational action in N.G.R.[4]

**) Latin indices are put in parentheses to discriminate them from Greek indices.

***) We will use the natural unit $\hbar = c = 1$.

2.2.2. *The gravitational and electromagnetic field equations in vacuum*

The electromagnetic Lagrangian density $L_{e.m.}$is given by****)

$$L_{e.m.} = -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \ . \tag{2·56}$$

We consider a system described by the Lagrangian density $L \stackrel{\text{def}}{=} L_G + L_{e.m.}$. The gravitational and electromagnetic field equations for this system are the following:

$$G^{\mu\nu}(\{\}) + K^{\mu\nu} = \kappa T^{\mu\nu}_{e.m.} \ , \tag{2·57}$$

$$\partial_\mu(\sqrt{-g} J^{kl\mu}) = 0 \ , \tag{2·58}$$

$$\partial_\nu \left(\sqrt{-g} F^{\mu\nu}\right) = 0 \ , \tag{2·59}$$

where we have defined

$$G_{\mu\nu}(\{\}) \stackrel{\text{def}}{=} R_{\mu\nu}(\{\}) - \frac{1}{2} g_{\mu\nu} R(\{\}) \ , \tag{2·60}$$

and

$$R_{\mu\nu}(\{\}) \stackrel{\text{def}}{=} R^{\lambda}{}_{\mu\lambda\nu}(\{\}) \ , \quad R(\{\}) \stackrel{\text{def}}{=} g^{\mu\nu} R_{\mu\nu}(\{\}) \tag{2·61}$$

with the Riemann-Christoffel curvature tensor

$$R^{\lambda}{}_{\rho\mu\nu}(\{\}) \stackrel{\text{def}}{=} 2 \left(\partial_{[\mu} \begin{Bmatrix} \lambda \\ \rho \ \nu] \end{Bmatrix} + \begin{Bmatrix} \lambda \\ \sigma \ [\mu \end{Bmatrix} \begin{Bmatrix} \sigma \\ \rho \ \nu] \end{Bmatrix} \right) \ . \tag{2·62}$$

Also, $T^{\mu\nu}_{e.m.}$ is the energy-momentum tensor of electromagnetic field:

$$T^{\mu\nu}_{e.m.} \stackrel{\text{def}}{=} F^{\mu\rho} F^{\nu\sigma} g_{\rho\sigma} + g^{\mu\nu} L_{e.m.} \ , \tag{2·63}$$

and the tensors $K^{\mu\nu}$ and $J^{kl\mu}$ are defined by

$$K^{\mu\nu} \stackrel{\text{def}}{=} \frac{\kappa}{\lambda} \left[\frac{1}{2} \{ \ \varepsilon^{\mu\rho\sigma\lambda} (T^{\nu}{}_{\rho\sigma} - T_{\rho\sigma}{}^{\nu}) + \varepsilon^{\nu\rho\sigma\lambda} (T^{\mu}{}_{\rho\sigma} - T_{\rho\sigma}{}^{\mu}) \ \} a_\lambda - \frac{3}{2} a^\mu a^\nu - \frac{3}{4} g^{\mu\nu} a^\lambda a_\lambda \right] \tag{2·64}$$

and

$$J^{kl\mu} \stackrel{\text{def}}{=} -\frac{3}{2} b^k{}_\rho b^l{}_\sigma \varepsilon^{\rho\sigma\mu\nu} a_\nu \ , \tag{2·65}$$

respectively, where we have put

$$\lambda \stackrel{\text{def}}{=} \frac{9\kappa}{4\kappa c_3 + 3} \ . \tag{2·66}$$

2.2.3. *An exact solution of gravitational and electromagnetic field equations with a charged rotating source*

An exact solution of the field equations (2·57), (2·58) and (2·59), which represents the gravitational and electromagnetic fields surrounding a charged rotating source, has

****) We will use Heaviside-Lorentz rationalized unit.

been given in Ref. 10. This will be described below. The vector fields $b^k{}_\mu$ and the electromagnetic potential A_μ have the expressions:

$$b^k{}_\mu = {}^{(0)}b^k{}_\mu + \frac{a}{2} l^k l_\mu - \frac{Q^2}{2} m^k m_\mu \,, \tag{2·67}$$

$$A_\mu = -\frac{q}{4\pi}\sqrt{Z}\, l_\mu \,, \tag{2·68}$$

where ${}^{(0)}b^k{}_\mu$ are the dual components of constant vierbeins and they are defined by ${}^{(0)}b^k{}_\mu \stackrel{\text{def}}{=} \delta^k{}_\mu$, The functions Z, l_μ, m_μ, l^k and m^k are given by

$$\begin{aligned}
Z &= \frac{N}{D} \,, \\
l_0 &= \sqrt{Z} \,, \quad l_\alpha = \frac{2\sqrt{Z}}{D + r^2 + h^2}\left[N x_\alpha \, + \frac{h^2 x^3 \delta^3{}_\alpha}{N} - \varepsilon_{\alpha\beta 3} x^\beta \right] \,, \\
m_\mu &= \frac{l_\mu}{\sqrt{N}} \,, \quad l^k = \delta^k{}_\mu \eta^{\mu\nu} l_\nu \,, \quad m^k = \delta^k{}_\mu \eta^{\mu\nu} m_\nu \,,
\end{aligned} \tag{2·69}$$

where D and N are given by

$$D = \sqrt{(r^2 - h^2)^2 + 4h^2 (x^3)^2} \,, \quad N = \frac{\sqrt{r^2 - h^2 + D}}{\sqrt{2}} \tag{2·70}$$

with $r \stackrel{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. In Eq.(2·69), $\varepsilon_{\alpha\beta\gamma}$ stand for three dimensional totally anti-symmetric tensor with $\varepsilon_{123} = 1$. Also, we have defined

$$a \stackrel{\text{def}}{=} \frac{\kappa m}{4\pi} \,, \quad Q \stackrel{\text{def}}{=} \frac{q}{4\pi}\sqrt{\frac{\kappa}{2}} \,, \quad h \stackrel{\text{def}}{=} -\frac{J}{m} \tag{2·71}$$

with m, J and q being the active gravitational mass, the absolute value of the angular momentum and the electromagnetic charge of the source body, respectively.

For the solution given by Eqs.(2·67) and (2·68), the axial vector part a_μ of the torsion tensor is vanishing:

$$a_\mu = 0 \,, \tag{2·72}$$

and the metric agrees with the charged Kerr metric in G.R.

The asymptotic forms of vierbeins and the electromagnetic vector potential for large r are given by

$$\begin{aligned}
b^a{}_\alpha &= \delta^a{}_\alpha + \frac{1}{2}\left(a - \frac{Q^2}{r}\right)\frac{n^a n_\alpha}{r} - \frac{a h n^\beta}{2r^2}\left(\varepsilon_{\alpha\beta 3} n^a + \varepsilon^a{}_{\beta 3} n_\alpha\right) \\
&\quad + O^a{}_\alpha\left(\frac{1}{r^3}\right) \,, \\
b^{(0)}{}_\alpha &= -\frac{n_\alpha}{2r}\left(a - \frac{Q^2}{r}\right) + \frac{ah}{2r^2}\varepsilon_{\alpha\beta 3} n^\beta + O_\alpha\left(\frac{1}{r^3}\right) \,, \\
b^a{}_0 &= \frac{n^a}{2r}\left(a - \frac{Q^2}{r}\right) - \frac{ah}{2r^2}\varepsilon^a{}_{\beta 3} n^\beta + O^a\left(\frac{1}{r^3}\right) \,, \\
b^{(0)}{}_0 &= 1 - \frac{1}{2r}\left(a - \frac{Q^2}{r}\right) + O\left(\frac{1}{r^3}\right)
\end{aligned} \tag{2·73}$$

and

$$A_0 = -\frac{q}{4\pi r} + O\left(\frac{1}{r^3}\right) , \quad A_\alpha = -\frac{q}{4\pi}\frac{x^\alpha}{r^2} + O_\alpha\left(\frac{1}{r^3}\right) , \tag{2·74}$$

respectively, which follow from Eqs.(2·67) and (2·68). Here, we have defined $n^\alpha \stackrel{\text{def}}{=} x^\alpha/r$, and $O\left(1/r^w\right)$ with positive w denotes a term such that $\lim_{r\to\infty} r^w O\left(1/r^w\right) =$ constant. *)

2.2.4. *Asymptotic form of* ψ^k

The space-time in this theory has vanishing curvature tensor. When the torsion tensor vanishes identically in addition, the space-time is the Minkowski space-time, for which, translational gauge potentials $A^k{}_\mu$ can be chosen to be zero and Eq.(2·3) is reduced to

$$b^k{}_\mu = \psi^k{}_{,\mu} \, . \tag{2·75}$$

For the solution given by Eqs.(2·67) and (2·68), we have **)

$$T^k{}_{\mu\nu} = O^k{}_{[\mu\nu]}\left(\frac{1}{r^2}\right) , \tag{2·76}$$

and the space-time asymptotically approach the Minkowski space-time for large r.

The above discussion and the consideration in Ref. 11 to introduce vierbeins suggest that ψ^k can be regarded as Minkowskian coordinate at spatial infinity, and we are naturally led to employ the following form of ψ^k:

$$\begin{aligned} \psi^k &= {}^{(0)}b^k{}_\mu x^\mu + {}^{(0)}\psi^k + O^k\left(\frac{1}{r^\beta}\right) \quad , \quad (\beta > 0) \, , \\ \psi^k{}_{,\mu} &= {}^{(0)}b^k{}_\mu + O^k{}_\mu\left(\frac{1}{r^{1+\beta}}\right) , \\ \psi^k{}_{,\mu\nu} &= O^k{}_{(\mu\nu)}\left(\frac{1}{r^2}\right) , \end{aligned} \tag{2·77}$$

where ${}^{(0)}\psi^k$ and β are constants.

§3. Equivalence relations for the case of the solution given by Eqs.(2·67) and (2·68)

In this section, on the basis of the discussion in Refs. 9 and 12, we examine energy-momentum, angular momentum and charge for the solution given in the preceding section.

3.1. *The case in which* $\{\psi^k, A^k{}_\mu, A_\mu\}$ *is employed as the set of independent field variables*

We understand that the Lagrangian density $\mathbf{L} \stackrel{\text{def}}{=} \sqrt{-g}L$ is a function of ψ^k, $A^k{}_\mu$, A_μ and of their derivatives. For this case, the generator M_k of *internal* translations

*) The symbols a and α in $O^a{}_\alpha(1/r^3), O_\alpha(1/r^3)$ and $O^a(1/r^3)$ are to show that these terms have indices as indicated.

**) The parenthesis [ ] in the suffix of $O^k{}_{[\mu\nu]}(1/r^2)$ means that this term is anti-symmetric with respect to μ , ν.

and the generator S_{kl} of *internal* Lorentz transformations are[7)] the dynamical energy-momentum and the total (=spin + orbital) angular momentum, respectively, and they are expressed as

$$M_k \stackrel{\rm def}{=} \int_\sigma {}^{\rm tot}\mathbf{T}_k{}^\mu d\sigma_\mu \tag{3·78}$$

and

$$S_{kl} \stackrel{\rm def}{=} \int_\sigma {}^{\rm tot}\mathbf{S}_{kl}{}^\mu d\sigma_\mu \ . \tag{3·79}$$

Here, σ is a space-like surface, and $d\sigma_\mu$ is the surface element on it:

$$d\sigma_\mu = \frac{1}{3!}\varepsilon_{\mu\nu\lambda\rho}dx^\nu \wedge dx^\lambda \wedge dx^\rho \ . \tag{3·80}$$

We have

$$\mathbf{F}_k{}^{\mu\nu} = \mathbf{F}^{(1)}{}_k{}^{\mu\nu} + \mathbf{F}^{(2)}{}_k{}^{\mu\nu} \tag{3·81}$$

with

$$\mathbf{F}^{(1)}{}_k{}^{\mu\nu} \stackrel{\rm def}{=} \frac{1}{\kappa\sqrt{-g}}b_{k\rho}\partial_\sigma\left\{(-g)\, g^{\rho[\mu}g^{\nu]\sigma}\right\} + \mathbf{Z}_k{}^{\mu\nu} \ , \tag{3·82}$$

$$\mathbf{F}^{(2)}{}_k{}^{\mu\nu} \stackrel{\rm def}{=} \frac{3}{2\lambda}\sqrt{-g}\,\varepsilon_k{}^{lmn}b^\mu{}_m b^\nu{}_n a_l \ , \tag{3·83}$$

$$\mathbf{Z}_k{}^{\mu\nu} \stackrel{\rm def}{=} \frac{\sqrt{-g}}{\kappa}\left\{ b^{[\mu}{}_k(b^{\nu]l}b^\lambda{}_{l.\lambda} - b^{\lambda l}\, b^{\nu]}{}_{l,\lambda}) + b^\lambda{}_k b^{[\mu}{}_l b^{\nu]l}{}_{,\lambda}\right\} \ , \tag{3·84}$$

as is known by using the explicit form of L_G.[6)] From Eqs.(2·17),(2·24),(2·28) and (3·81), ${}^{\rm tot}\mathbf{S}_{kl}{}^\mu$ defined by Eq.(2·29) can be rewritten as

$$\begin{aligned}{}^{\rm tot}\mathbf{S}_{kl}{}^\mu = {} & 2\partial_\nu(\psi_{[k}\mathbf{F}_{l]}{}^{\mu\nu}) + \frac{2}{\kappa}\partial_\nu\left(\sqrt{-g}b^\mu{}_{[k}b^\nu{}_{l]} - {}^{(0)}b^\mu{}_{[k}{}^{(0)}b^\nu{}_{l]}\right) \\ & -b_{[k\nu}\mathbf{F}^{(2)}{}_{l]}{}^{\mu\nu} + 2\psi_{[k}\frac{\delta\mathbf{L}}{\delta A^{l]}{}_\mu} \ ,\end{aligned} \tag{3·85}$$

where ${}^{(0)}b^\mu{}_k$ are the components of the constant vierbeins: ${}^{(0)}b^\mu{}_k \stackrel{\rm def}{=} \delta^\mu{}_k$. For the vierbeins given by Eq.(2·67), we have

$$\mathbf{Z}_k{}^{\mu\nu} = 0 \ , \quad \mathbf{F}^{(2)}{}_k{}^{\mu\nu} = 0 \ , \tag{3·86}$$

and hence

$$\mathbf{F}_k{}^{\mu\nu} = \mathbf{F}^{(1)}{}_k{}^{\mu\nu} = \frac{1}{\kappa\sqrt{-g}}b_{k\rho}\partial_\sigma\left\{(-g)\, g^{\rho[\mu}g^{\nu]\sigma}\right\} \ . \tag{3·87}$$

3.1.1. *Energy-momentum*

When the field equation $\delta\mathbf{L}/\delta A^k{}_\mu = 0$ is satisfied, Eq.(3·78) can be rewritten as

$$M_k = \int_\sigma \partial_\nu\mathbf{F}_k{}^{\mu\nu}d\sigma_\mu = \int_S \mathbf{F}_k{}^{0\alpha}r^2 n_\alpha d\Omega \ , \tag{3·88}$$

by using the identity (2·24). Here, S and $d\Omega$ stand for the two dimensional surface of σ and the differential solid angle, respectively. Equations (3·81),(3·86),(3·87),(3·88) and (A·1) give

$$M_{(0)} = -m \ , \quad M_a = 0 \ . \tag{3·89}$$

The quantity M_k is the total energy-momentum vector of the system. The first of Eq.(3·89) shows the *equality* of the active gravitational mass and the inertial mass.

3.1.2. *Angular momentum*

From Eqs.(3·79) and (3·85), the total angular momentum is expressed as

$$S_{kl} = 2\int_\sigma \partial_\nu \left[\psi_{[k}\mathbf{F}_{l]}{}^{\mu\nu} + \frac{1}{\kappa}\left\{ \sqrt{-g}\, b^\mu{}_{[k}\, b^\nu{}_{l]} \;-\; {}^{(0)}b^\mu{}_{[k}{}^{(0)}b^\nu{}_{l]} \right\} \right] d\sigma_\mu \ , \tag{3·90}$$

from which the expression

$$\begin{aligned} S_{(0)a} &= {}^{(0)}\psi_a m = {}^{(0)}\psi_a M_{(0)} - {}^{(0)}\psi_{(0)} M_a \ , \\ S_{ab} &= J\varepsilon_{ab3} = J\varepsilon_{ab3} + {}^{(0)}\psi_a M_b - {}^{(0)}\psi_b M_a \end{aligned} \tag{3·91}$$

is obtained by the use of Eqs.(2·73),(2·77), (3·78), (3·81), (3·87) and (A·1).

In the above, the terms of the form ${}^{(0)}\psi_k M_l - {}^{(0)}\psi_l M_k$ are regarded as the orbital angular momentum around the origin of the internal space and they are conserved by themselves.[12] Equation (3·91) implies that the total angular momentum is equal to the angular momentum of the rotating source.

3.1.3. *Canonical energy-momentum and "extended orbital angular momentum"*

The generator $M^c{}_\mu$ of coordinate translations and the generator $L_\mu{}^\nu$ of $GL(4, \mathbf{R})$ coordinate transformations are the canonical energy-momentum and the "extended orbital angular momentum", *) respectively. They have the expressions:

$$M^c{}_\mu \stackrel{\text{def}}{=} \int_\sigma \tilde{\mathbf{T}}_\mu{}^\nu d\sigma_\nu = \int_\sigma \partial_\tau \mathbf{\Psi}_\mu{}^{\nu\tau} d\sigma_\nu \ , \tag{3·92}$$

$$L_\mu{}^\nu \stackrel{\text{def}}{=} \int_\sigma \mathbf{M}_\mu{}^{\nu\lambda} d\sigma_\lambda = -2\int_\sigma \partial_\tau (x^\nu \mathbf{\Psi}_\mu{}^{\lambda\tau}) d\sigma_\lambda \ . \tag{3·93}$$

Asymptotic behaviors of translational gauge potentials $A^k{}_\mu$ at spatial infinity are given as follows:

$$\begin{aligned} A^{(0)}{}_0 &= -\frac{a}{2r} + O\left(\frac{1}{r^{1+\beta}}\right) \ , \quad A^{(0)}{}_\alpha = -\frac{a}{2r} n_\alpha + O_\alpha\left(\frac{1}{r^{1+\beta}}\right) \ , \\ A^a{}_0 &= \frac{a}{2r} n^a + O^a\left(\frac{1}{r^{1+\beta}}\right) \ , \quad A^a{}_\alpha = \frac{a}{2r} n^a n_\alpha + O^a{}_\alpha\left(\frac{1}{r^{1+\beta}}\right) \ , \end{aligned} \tag{3·94}$$

which are known from Eqs.(2·17),(2·73) and (2·77). Then $M^c{}_\mu$ vanishes and trivial:

$$M^c{}_\mu = \int_S \mathbf{F}_k{}^{0\alpha} A^k{}_\mu r^2 n_\alpha d\Omega = 0 \ , \tag{3·95}$$

while $L_\mu{}^\nu$ is expressed as

$$L_\mu{}^\nu = -2\int_S x^\nu (\mathbf{F}_k{}^{0\alpha} A^k{}_\mu + \mathbf{F}^{0\alpha} A_\mu) r^2 n_\alpha d\Omega \ , \tag{3·96}$$

and the non-zero components are given by

$$L_1{}^1 = L_2{}^2 = L_3{}^3 = \frac{q^2}{6\pi} \ . \tag{3·97}$$

These are shown by the use of Eqs.(2·36),(2·74),(3·81), (3·94) and (A·1). Thus, the orbital angular momentum $L_{[\mu\nu]}$ is vanishing: $L_{[\mu\nu]} = 0$.

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*) Note that the anti-symmetric part $L_{[\mu\nu]} \stackrel{\text{def}}{=} \eta_{[\nu\lambda} L_{\mu]}{}^\lambda$ is the orbital angular momentum.

3.1.4. *Charge*

The charge is defined as the generator of $U(1)$ gauge transformations and is given by

$$C \stackrel{\text{def}}{=} \int_\sigma \partial_\nu \left(\frac{\partial \mathbf{L}}{\partial A_{\mu,\nu}} \right) d\sigma_\mu = \int_\sigma \partial_\nu (\sqrt{-g} F^{\mu\nu}) d\sigma_\mu = -\int_S (\partial^\alpha A^0) n_\alpha r^2 d\Omega = q \ , \qquad (3\cdot98)$$

where we have used Eq.(2·74). This means the *equality* of the total charge of the system and the charge of the source.

3.2. *The case in which $\{\psi^k, b^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables*

Let us denote $\boldsymbol{L}$ and L expressed as a function of $\psi^k, b^k{}_\mu, A_\mu$ and of their derivatives by $\hat{\mathbf{L}}$ and $\hat{L}$, respectively. The action $\mathbf{I}$ is now written as

$$\hat{\mathbf{I}} = \int_{\mathcal{D}} \hat{L} dv = \mathbf{I} \ . \qquad (3\cdot99)$$

Various identities are derived from the requirements (R.i) and (R.ii), among which we have

$$\frac{\delta \hat{\mathbf{L}}}{\delta \psi^k} \equiv 0 \ , \qquad (3\cdot100)$$

$$^{\text{tot}}\hat{\mathbf{T}}_k{}^\mu \equiv 0 \ , \qquad (3\cdot101)$$

$$\partial_\mu {}^{\text{tot}}\hat{\mathbf{S}}_{kl}{}^\mu - 2\frac{\delta \hat{\mathbf{L}}}{\delta \psi^{[k}} \psi_{l]} - 2\frac{\delta \hat{\mathbf{L}}}{\delta b^{[k}{}_\mu} b_{l]\mu} \equiv 0 \ , \qquad (3\cdot102)$$

$$\hat{\tilde{\mathbf{T}}}_\mu{}^\nu - \partial_\lambda \hat{\boldsymbol{\Psi}}_\mu{}^{\nu\lambda} - \frac{\delta \hat{\mathbf{L}}}{\delta b^k{}_\nu} b^k{}_\mu - \frac{\delta \hat{\mathbf{L}}}{\delta A_\nu} A_\mu \equiv 0 \ , \qquad (3\cdot103)$$

where we have defined

$$\hat{\mathbf{L}} \stackrel{\text{def}}{=} \sqrt{-g} \hat{L} \ , \qquad (3\cdot104)$$

$$^{\text{tot}}\hat{\mathbf{T}}_k{}^\mu \stackrel{\text{def}}{=} \frac{\partial \hat{\mathbf{L}}}{\partial \psi^k{}_{,\mu}} \ , \qquad (3\cdot105)$$

$$^{\text{tot}}\hat{\mathbf{S}}_{kl}{}^\mu \stackrel{\text{def}}{=} -2\frac{\partial \hat{\mathbf{L}}}{\partial \psi^{[k}{}_{,\mu}} \psi_{l]} - 2\hat{\mathbf{F}}_{[k}{}^{\nu\mu} b_{l]\nu} \ , \qquad (3\cdot106)$$

$$\hat{\tilde{\mathbf{T}}}_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu{}^\nu \hat{\mathbf{L}} - \hat{\mathbf{F}}_k{}^{\lambda\nu} b^k{}_{\lambda,\mu} - \hat{\mathbf{F}}^{\lambda\nu} A_{\lambda,\mu} - \frac{\partial \hat{\mathbf{L}}}{\partial \psi^k{}_{,\nu}} \psi^k{}_{,\mu} \ , \qquad (3\cdot107)$$

$$\hat{\boldsymbol{\Psi}}_\lambda{}^{\mu\nu} \stackrel{\text{def}}{=} \hat{\mathbf{F}}_k{}^{\mu\nu} b^k{}_\lambda + \hat{\mathbf{F}}^{\mu\nu} A_\lambda = -\hat{\boldsymbol{\Psi}}_\lambda{}^{\nu\mu} \ , \qquad (3\cdot108)$$

$$\hat{\mathbf{F}}_k{}^{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \hat{\mathbf{L}}}{\partial b^k{}_{\mu,\nu}} = \mathbf{F}_k{}^{\mu\nu} \ , \quad \hat{\mathbf{F}}^{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \hat{\mathbf{L}}}{\partial A_{\mu,\nu}} = \mathbf{F}^{\mu\nu} \ . \qquad (3\cdot109)$$

From Eqs.(3·100) and (3·102), we see that

$$\partial_\mu {}^{\text{tot}}\hat{\mathbf{S}}_{kl}{}^\mu = 0 \ , \qquad (3\cdot110)$$

when the field equations of $b^k{}_\mu$ are satisfied. From Eqs.(3·103) and (3·108), it follows that

$$\partial_\nu \hat{\tilde{\mathbf{T}}}_\mu{}^\nu = 0 \ , \tag{3·111}$$

$$\partial_\nu \hat{\mathbf{M}}_\lambda{}^{\mu\nu} = 0 \ . \tag{3·112}$$

when the field equations $\delta\hat{\mathbf{L}}/\delta b^k{}_\mu = 0$ and $\delta\hat{\mathbf{L}}/\delta A_\mu = 0$ are both satisfied, where $\hat{\mathbf{M}}_\lambda{}^{\mu\nu} \stackrel{\text{def}}{=} 2(\hat{\mathbf{\Psi}}_\lambda{}^{\mu\nu} - x^\mu \hat{\tilde{\mathbf{T}}}_\lambda{}^\nu)$. Equations (3·110), (3·111) and (3·112) are the differential conservation laws of the "spin" angular momentum, of the canonical angular momentum and of the "extended orbital angular momentum", respectively.

The density ${}^{\text{tot}}\hat{\mathbf{S}}_{kl}{}^\mu$ defined by Eq.(3·106) can be rewritten as

$${}^{\text{tot}}\hat{\mathbf{S}}_{kl}{}^\mu = \frac{2}{\kappa}\partial_\nu \left(\sqrt{-g} b^\mu{}_{[k} b^\nu{}_{l]} - {}^{(0)}b^\mu{}_{[k}{}^{(0)}b^\nu{}_{l]}\right) - b_{[k\nu}\mathbf{F}^{(2)}{}_{l]}{}^{\mu\nu} \ , \tag{3·113}$$

by the use of Eqs.(3·81), (3·101), (3·109) and (3·105).

3.2.1. *Energy-momentum*

The dynamical energy-momentum $\hat{M}_k$, which is the generator of *internal* translations, vanishes identically

$$\hat{M}_k \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\hat{\mathbf{T}}_k{}^\mu d\sigma_\mu \equiv 0 \ , \tag{3·114}$$

as is evident from Eq.(3·101).

3.2.2. *Spin angular momentum*

The generator $\hat{S}_{kl}$ of internal Lorentz transformations is expressed as

$$\hat{S}_{kl} \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\hat{\mathbf{S}}_{kl}{}^\mu d\sigma_\mu = \frac{2}{\kappa}\int_S \left(\sqrt{-g}\, b^0{}_{[k} b^\alpha{}_{l]} - {}^{(0)}b^0{}_{[k}{}^{(0)}b^\alpha{}_{l]}\right) n_\alpha r^2 d\Omega \ , \tag{3·115}$$

as is shown by using Eq.(3·113), and we obtain

$$\hat{S}_{(0)a} = 0 \ , \quad \hat{S}_{ab} = \frac{1}{3} J\varepsilon_{ab3} \tag{3·116}$$

by using Eq.(2·73).

3.2.3. *Canonical energy-momentum and "extended orbital angular momentum"*

The generator $\hat{M}^c{}_\mu$ of coordinate translations and the generator $\hat{L}_\mu{}^\nu$ of $GL(4,\mathbf{R})$ coordinate transformations are the canonical energy-momentum and the "extended orbital angular momentum", respectively. They have the expressions:

$$\hat{M}^c{}_\mu \stackrel{\text{def}}{=} \int_\sigma \hat{\tilde{\mathbf{T}}}_\mu{}^\nu \sigma_\nu = \int_\sigma \partial_\tau \hat{\mathbf{\Psi}}_\mu{}^{\nu\tau} d\sigma_\nu \ , \tag{3·117}$$

$$\hat{L}_\mu{}^\nu \stackrel{\text{def}}{=} \int_\sigma \hat{\mathbf{M}}_\mu{}^{\nu\lambda} d\sigma_\lambda = -2\int_\sigma \partial_\tau \left(x^\nu \hat{\mathbf{\Psi}}_\mu{}^{\lambda\tau}\right) d\sigma_\lambda \ . \tag{3·118}$$

Then, we have

$$\hat{M}^c{}_0 = -m \ , \quad \hat{M}^c{}_\alpha = 0 \ . \tag{3·119}$$

Thus, $\hat{M}^c{}_\mu$ is the total energy-momentum and the *equality* of the active gravitational mass and the inertial mass holds. Also, $\hat{L}_\mu{}^\nu$ is given by

$$\begin{aligned} &\hat{L}_0{}^0 = 2x^0 m\ , \quad \hat{L}_0{}^\alpha = 0\ , \quad \hat{L}_\alpha{}^0 = 0\ , \\ &\hat{L}_\alpha{}^\beta = \frac{2}{3} J \varepsilon_\alpha{}^{\beta 3}\ , \ \ (\alpha \neq \beta)\ , \quad \hat{L}_1{}^1 = \hat{L}_2{}^2 = \hat{L}_3{}^3 = \infty\ . \end{aligned} \tag{3·120}$$

Equations (3·119) and (3·120) are obtained by using Eqs.(2·73),(2·74),(3·87),(3·108), (3·109) and (A·1).

The orbital angular momentum $\hat{L}_{[\mu\nu]}$ is given by:

$$\hat{L}_{[0\alpha]} = 0\ , \quad \hat{L}_{[\alpha\beta]} = \frac{2}{3} J \varepsilon_{\alpha\beta 3}\ . \tag{3·121}$$

If we define the total angular momentum of the system by[9)]

$$\hat{J}_{kl} \stackrel{\mathrm{def}}{=} \hat{S}_{kl} + {}^{(0)}b^\mu{}_k {}^{(0)}b^\nu{}_l \hat{L}_{[\mu\nu]}\ , \tag{3·122}$$

then we have

$$\hat{J}_{(0)k} = 0\ , \quad \hat{J}_{ab} = J \varepsilon_{ab3}\ , \tag{3·123}$$

and the total angular momentum is equal to the angular momentum of the rotating source.

3.2.4. *Charge*

The generator $\hat{C}$ of $U(1)$ gauge transformations is given by

$$\hat{C} \stackrel{\mathrm{def}}{=} \int_\sigma \partial_\nu \left(\frac{\partial \hat{\mathbf{L}}}{\partial A_{\mu,\nu}} \right) d\sigma_\mu = q\ , \tag{3·124}$$

which means the *equality* of the total charge of the system and the charge of the source.

§4. Restrictions imposed on field variables by generalized equivalence principle

In the preceding section, we have examined the solution given by Eqs.(2·67) and (2·68), and the results show that the total energy-momentum, the total angular momentum and the total charge of the system, which are generators of transformations, are equal to the corresponding active quantities of central gravitating body. The total mass, which is equal to the total energy divided by the square of velocity of light, can be regarded as the inertial mass of the system. Thus, the results include the equality of the inertial mass and the active gravitational mass, which implies that an equivalence principle is satisfied by this solution.

In view of the above, we regard, total momentum, total angular momentum and total charge as “inertial” quantities, and say that a generalized equivalence principle (G.E.P.) is satisfied, if total energy-momentum, total angular momentum and total charge are all equal to corresponding quantities of the source.

The axial vector part a_μ vanishes for our solution as stated previously, and the field equations (2·57), (2·58) and (2·59) are covariant under general coordinate transformations and under local Lorentz transformations which keep a_μ vanishing. Thus, new solutions are obtainable by applying the general coordinate transformations and the restricted local Lorentz transformations $b^{k\prime}{}_{\mu} = A^{k}{}_{l}(x) b^{l}{}_{\mu}$ which satisfy the condition:[4),*)]

$$\varepsilon^{\mu\nu\rho\sigma} b^{k}{}_{\nu} b^{l}{}_{\rho} A^{m}{}_{k}(x) A_{ml}(x)_{,\sigma} = 0 \tag{4·125}$$

to the solution given by Eqs.(2·67) and (2·68).

In this section, we examine restrictions imposed on solutions by the requirement that G.E.P. is satisfied.

We look for new solutions having suitable asymptotic behaviors by considering the following $\overline{\text{Poincaré}}$ gauge transformations:

(1) Local $SL(2.C)$ transformation

$$H^{k}{}_{l} \stackrel{\mathrm{def}}{=} \left(\varLambda(a^{-1})\right)^{k}{}_{l} = \varLambda^{k}{}_{l} + \varLambda^{k}{}_{m} \omega^{m}{}_{l}(x) \ . \tag{4·126}$$

(2) Local internal translation

$$t^{k} = {}^{(0)}t^{k} + b^{k}(x) \ . \tag{4·127}$$

Here, $\varLambda^{k}{}_{l}$ and ${}^{(0)}t^{k}$ denote a constant internal Lorentz transformation and the constant internal translation, respectively, and $\omega^{k}{}_{l}$ and b^{k} are functions satisfying the following conditions:

$$\omega^{k}{}_{l}(x) = O^{k}{}_{l}\left(\frac{1}{r^{p}}\right) \ , \quad \omega^{k}{}_{l,\mu}(x) = O^{k}{}_{l\mu}\left(\frac{1}{r^{p+1}}\right) \quad , \ (p>0) \ , \tag{4·128}$$

$$b^{k}(x) = O^{k}\left(\frac{1}{r^{\gamma}}\right) \ , \quad b^{k}{}_{,\mu}(x) = O^{k}{}_{\mu}\left(\frac{1}{r^{\gamma+1}}\right) \quad , \ (\gamma>0) \ . \tag{4·129}$$

The transformation $H^{k}{}_{l} \stackrel{\mathrm{def}}{=} \left(\varLambda(a^{-1})\right)^{k}{}_{l}$ is a Lorentz transformation satisfying Eq.(4·125), if and only if

$$\omega_{kl} + \omega_{lk} + \omega_{mk}\omega^{m}{}_{l} = 0 \ , \tag{4·130}$$

$$\varepsilon^{\mu\nu\lambda\rho}(\omega_{kl,\rho} + \omega_{mk}\omega^{m}{}_{l,\rho}) b^{k}{}_{\nu} b^{l}{}_{\lambda} = 0 \ , \tag{4·131}$$

are both satisfied. The conditions (4·130) and (4·131) are equivalent to

$$\omega_{\mu\nu} + \omega_{\nu\mu} + \omega_{\lambda\mu}\omega^{\lambda}{}_{\nu} = 0 \tag{4·132}$$

and

$$X_{\mu\nu\lambda} + X_{\lambda\mu\nu} + X_{\nu\lambda\mu} = 0 \ , \tag{4·133}$$

respectively, where we have defined

$$\omega_{\mu\nu} \stackrel{\mathrm{def}}{=} b^{k}{}_{\mu} b^{l}{}_{\nu} \omega_{kl} \ , \tag{4·134}$$

$$\begin{aligned} X_{\mu\nu\lambda} \stackrel{\mathrm{def}}{=} {} & \omega_{\mu\nu,\lambda} + \omega^{\tau}{}_{\mu}\omega_{\tau\nu,\lambda} - b^{\tau}{}_{k} b^{k}{}_{\mu,\lambda}\omega_{\tau\nu} - b^{\tau}{}_{k} b^{k}{}_{\nu,\lambda}\omega_{\mu\tau} \\ & + b^{\tau k} b^{\rho}{}_{k,\lambda}\omega_{\tau\mu}\omega_{\rho\nu} + b^{k}{}_{\nu} b^{\tau}{}_{k,\lambda}\omega^{\rho}{}_{\mu}\omega_{\rho\tau} \ . \end{aligned} \tag{4·135}$$

The function $X_{\mu\nu\lambda}$ is anti-symmetric with respect to the first two indices:

$$X_{\mu\nu\lambda} = -X_{\nu\mu\lambda} \ . \tag{4·136}$$

---

*) Note that a_{μ} is invariant under the local Lorentz transformation $A^{k}{}_{l}(x)$ if and only if this condition is satisfied.

From Eqs.(4·128), (4·132) and (4·133), $\omega_{\mu\nu}$ is known to have the expression

$$\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + f_{\mu\nu}(x) \tag{4·137}$$

with

$$\begin{aligned} \partial_\mu \omega_\nu - \partial_\nu \omega_\mu &= O_{[\mu\nu]} \left(\frac{1}{r^p} \right) \ , \\ f_{\mu\nu}(x) &= O_{\mu\nu} \left(\frac{1}{r^s} \right) \ , \quad (p < s) \ . \end{aligned} \tag{4·138}$$

In addition, we require the leading term of ω_{kl} at spatial infinity to be spherically symmetric, then we can put

$$\omega^0 = A(r, x^0) \quad , \quad \omega^\alpha = n^\alpha B(r, x^0) \tag{4·139}$$

with certain functions A , B of r and of x^0.

Also, we consider the following coordinate transformation:

$$\begin{aligned} x^{\mu\prime} &= C^\mu{}_\nu x^\nu + D^\mu(x) \ , \\ \frac{\partial x^{\mu\prime}}{\partial x^\nu} &= C^\mu{}_\nu + a^\mu{}_\nu(x) \ , \\ a^\mu{}_\nu &\stackrel{\text{def}}{=} D^\mu{}_{,\nu} \ , \\ a^\mu{}_\nu(x) &= O^\mu{}_\nu \left(\frac{1}{r^u} \right) \quad , \quad a^\mu{}_{\nu,\lambda} = O^\mu{}_{\nu\lambda} \left(\frac{1}{r^{u+1}} \right) \quad , \quad (u > 0) \ , \end{aligned} \tag{4·140}$$

where $C^\mu{}_\nu$ denote constant Lorentz transformation, and $D^\mu(x)$ satisfies the following condition:

$$\lim_{r \to \infty} \frac{D^\mu(x)}{r} = 0 \ . \tag{4·141}$$

We write $\partial x^\mu / \partial x^{\nu\prime}$ as follows:

$$\frac{\partial x^\mu}{\partial x^{\nu\prime}} = \left(C^{-1} \right)^\mu{}_\nu + d^\mu{}_\nu(x) \tag{4·142}$$

with $(C^{-1})^\mu{}_\nu$ being constants satisfying $(C^{-1})^\mu{}_\lambda C^\lambda{}_\nu = \delta^\mu{}_\nu$.

The vierbeins and vector potentials given by

$$b^{k\prime}{}_{\mu\prime} \stackrel{\text{def}}{=} H^k{}_l \frac{\partial x^\nu}{\partial x^{\mu\prime}} b^l{}_\nu \ , \quad A_{\mu\prime} \stackrel{\text{def}}{=} \frac{\partial x^\nu}{\partial x^{\mu\prime}} A_\nu \tag{4·143}$$

with $b^k{}_\mu$ and A_μ given by Eqs.(2·67) and (2·68) are solutions of gravitational and electromagnetic field equations. This is so irrespective of the values of the parameters p , s , β , γ and u. G.E.P. is considered to be satisfied if the energy-momentum, the angular momentum and the charge are all have correct transformation properties which their indices indicate. But, this is not necessarily satisfied for arbitrary values of these parameters, i.e. there are solutions which do not satisfy G.E.P.

We examine restrictions imposed on these parameters by the requirement that G.E.P. is satisfied.

4.1. *The case in which $\{\psi^k, A^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables*

Under the combined transformation of the $\overline{\text{Poincaré}}$ gauge transformation given by Eqs.(4·126) and (4·127) and satisfying the conditions (4·130) and (4·131) and of the coordinate transformation (4·140), $\mathbf{F}^{(1)}{}_k{}^{\mu\nu}, \mathbf{F}^{(2)}{}_k{}^{\mu\nu}$ and $\mathbf{F}^{\mu\nu}$ transform according as

$$\begin{aligned}\mathbf{F}^{(1)}{}_{k'}{}^{\mu'\nu'} &= \Delta \frac{\partial x^{\mu'}}{\partial x^\rho}\frac{\partial x^{\nu'}}{\partial x^\sigma} H_k{}^l \mathbf{F}^{(1)}{}_l{}^{\rho\sigma} \\ &\quad + \frac{\Delta}{\kappa} U_k{}^{lmn}{}_\lambda W^{\rho\sigma\lambda}{}_{lmn} \frac{\partial x^{\mu'}}{\partial x^\rho}\frac{\partial x^{\nu'}}{\partial x^\sigma} \,, \qquad (4·144)\end{aligned}$$

$$\mathbf{F}^{(2)}{}_{k'}{}^{\mu'\nu'} = 0 = \Delta \frac{\partial x^{\mu'}}{\partial x^\rho}\frac{\partial x^{\nu'}}{\partial x^\sigma} H_k{}^l \mathbf{F}^{(2)}{}_l{}^{\rho\sigma} \,, \qquad (4·145)$$

$$\mathbf{F}^{\mu'\nu'} = \Delta \frac{\partial x^{\mu'}}{\partial x^\rho}\frac{\partial x^{\nu'}}{\partial x^\sigma} \mathbf{F}^{\rho\sigma} \,, \qquad (4·146)$$

where we have defined *)

$$U_k{}^{lmn}{}_\lambda \stackrel{\text{def}}{=} H_k{}^l V^{mn}{}_\lambda \,, \qquad (4·147)$$

$$V^{mn}{}_\lambda \stackrel{\text{def}}{=} H^{lm} H_l{}^n{}_{,\lambda} = -V^{nm}{}_\lambda \,, \qquad (4·148)$$

$$W^{\mu\nu\lambda}{}_{klm} \stackrel{\text{def}}{=} b^{[\mu}{}_k b^{\nu]}{}_l b^\lambda{}_m + b^{[\mu}{}_m b^{\nu]}{}_k b^\lambda{}_l + b^{[\mu}{}_l b^{\nu]}{}_m b^\lambda{}_k \,, \qquad (4·149)$$

$$\Delta \stackrel{\text{def}}{=} \det\left(\frac{\partial x^\mu}{\partial x^{\nu'}}\right) . \qquad (4·150)$$

Equations (4·144),(4·145) and (4·146) show that $\mathbf{F}^{(1)}{}_k{}^{\mu\nu}$,$\mathbf{F}^{(2)}{}_k{}^{\mu\nu}$ and $\mathbf{F}^{\mu\nu}$ are transformed as tensor densities under coordinate transformations. The function $W^{\mu\nu\lambda}{}_{klm}$ is totally anti-symmetric both in upper indices and in lower indices.

4.1.1. *Energy-momentum*

From Eqs.(3·88) and (4·144), M_k is known to transform according as**)

$$M_{k'} \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\mathbf{T}_{k'}{}^{\mu'} d\sigma_{\mu'} = \int_\sigma \partial_{\nu'} \mathbf{F}_{k'}{}^{\mu'\nu'} d\sigma_{\mu'} = \Lambda_k{}^l M_l + \frac{3}{\kappa}\int U_k{}^{[0\alpha\beta]}{}_\beta n_\alpha r^2 d\Omega \,. \quad (4·151)$$

The energy-momentum M_k obeys the correct transformation rule

$$M_{k'} = \Lambda_k{}^l M_l \,, \qquad (4·152)$$

if the condition

$$p > \frac{1}{2}\,, \quad s > 1 \qquad (4·153)$$

is satisfied. ***)

*) For simplicity, we restrict our considerations to the case with $\Delta > 0$. An extension to the case with arbitrary non-vanishing Δ can be made without difficulty.

**) $A^{[\mu\nu\lambda]} \stackrel{\text{def}}{=} \frac{1}{3}\{A^{\mu[\nu\lambda]} + A^{\nu[\lambda\mu]} + A^{\lambda[\mu\nu]}\}$.

***) For derivations of the conditions (4·153),(4·156),(4·157),(4·158),(4·164),(4·172) and (4·176), elementary but rather tedious calculations are needed. In Appendix A, we give lists of asymptotic forms of $\mathbf{F}^{(1)}{}_k{}^{\mu\nu}$,$V^{mn}{}_\lambda$ and $W^{\mu\nu\lambda}{}_{klm}$ for large r, which are useful in calculations.

4.1.2. *Angular momentum*

From Eqs.(3·90) and (4·144), S_{kl} is known to transform according as

$$
\begin{aligned}
S_{k\prime l\prime} &\stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\mathbf{S}_{k\prime l\prime}{}^{\mu\prime} d\sigma_{\mu\prime} \\
&= 2\int_\sigma \partial_{\nu\prime}\left[\psi_{[k\prime}\mathbf{F}_{l\prime]}{}^{\mu\prime\nu\prime} + \frac{1}{\kappa}\left(\sqrt{-g\prime}b^{\mu\prime}{}_{[k\prime}b^{\nu\prime}{}_{l\prime]} - {}^{(0)}b^{\mu\prime}{}_{[k\prime}{}^{(0)}b^{\nu\prime}{}_{l\prime]}\right)\right] d\sigma_{\mu\prime} \\
&= \Lambda_k{}^m \Lambda_l{}^n (S_{mn} - 2^{(0)}t_{[m}M_{n]}) \\
&\quad +2\Lambda_k{}^m \int \{\omega_m{}^n(\psi_n - t_n) - b_m(x)\} H_l{}^n \mathbf{F}_n{}^{0\alpha} n_\alpha r^2 d\Omega \\
&\quad +\frac{2}{\kappa}\int H_{[k}{}^m H_{l]}{}^n(\psi_m - t_m)H^{ij}H_i{}^h{}_{,\beta}W^{0\alpha\beta}{}_{njh} n_\alpha r^2 d\Omega \\
&\quad +\frac{2}{\kappa}\int H_{[k}{}^m(H_{l]}{}^n + \Lambda_{l]}{}^m\omega_m{}^n)(b^0{}_{[m}b^\alpha{}_{n]} - {}^{(0)}b^0{}_{[m}{}^{(0)}b^\alpha{}_{n]}) n_\alpha r^2 d\Omega\ ,
\end{aligned} \tag{4·154}
$$

where we have defined ${}^{(0)}b^{\mu\prime}{}_{k\prime} \stackrel{\text{def}}{=} \lim_{r\to\infty} H_k{}^l(\partial x^{\mu\prime}/\partial x^\nu)b^\nu{}_l = \Lambda_k{}^l C^\mu{}_\nu{}^{(0)}b^\nu{}_l$. The angular momentum obeys the correct transformation rule: *)

$$
S_{k\prime l\prime} = \Lambda_k{}^m \Lambda_l{}^n (S_{mn} - 2^{(0)}t_{[m}M_{n]})\ , \tag{4·155}
$$

if the following conditions are satisfied:

$$
p > \frac{1}{2}\ , \quad s > 2\ . \tag{4·156}
$$

and

$$
\left[\{\, p+\beta > 1, \quad p+\gamma > 1\} \ \text{or} \ \left\{ O^{(0)}\left(\frac{1}{r^\beta}\right) = f(r,x^0)\ ,\ O^{(0)}\left(\frac{1}{r^\gamma}\right) = g(r,x^0) \right\}\right] \tag{4·157}
$$

and

$$
\left[\{\, p+\beta > 1\ , \quad p+\gamma > 1\} \ \text{or} \ \left\{ O^a\left(\frac{1}{r^\beta}\right) = n^a h(r,x^0)\ ,\ O^a\left(\frac{1}{r^\gamma}\right) = n^a k(r,x^0) \right\}\right] \tag{4·158}
$$

with f, g, h and k being some functions of r and of x^0, where the terms $O^k(1/r^\beta)$ and $O^k(1/r^\gamma)$ are those in Eq.(2·77) and in Eq.(4·129), respectively.

4.1.3. *Canonical energy-momentum and "extended orbital angular momentum"*

The transformed canonical energy-momentum $M^c{}_{\mu\prime}$ and the "extended orbital angular momentum" $L_{\mu\prime}{}^{\nu\prime}$ are given by

$$
\begin{aligned}
M^c{}_{\mu\prime} &\stackrel{\text{def}}{=} \int_\sigma \tilde{\mathbf{T}}_{\mu\prime}{}^{\nu\prime} d\sigma_{\nu\prime} = \int_\sigma \partial_{\lambda\prime}\mathbf{\Psi}_{\mu\prime}{}^{\nu\prime\lambda\prime} d\sigma_{\nu\prime} \\
&= \int \mathbf{\Psi}_{\mu\prime}{}^{\nu\prime\lambda\prime} J \frac{\partial x^0}{\partial x^{\nu\prime}}\frac{\partial x^\alpha}{\partial x^{\lambda\prime}} n_\alpha r^2 d\Omega
\end{aligned} \tag{4·159}
$$

*) The term $2^{(0)}t_{[k}M_{l]}$ comes from the translation (4·127), and it is an additional orbital angular momentum around the origin of *internal* space.

$$L_{\mu'}{}^{\nu'} \stackrel{\rm def}{=} \int_\sigma \mathbf{M}_{\mu'}{}^{\nu'\lambda'} d\sigma_{\lambda'} = -2\int_\sigma \partial_{\tau'}(x^{\nu'}\mathbf{\Psi}_{\mu'}{}^{\lambda'\tau'})d\sigma_{\lambda'}$$
$$= -2\int x^{\nu'}\mathbf{\Psi}_{\mu'}{}^{\lambda'\tau'} J \frac{\partial x^0}{\partial x^{\lambda'}}\frac{\partial x^\alpha}{\partial x^{\tau'}} n_\alpha r^2 d\Omega \qquad (4\cdot160)$$

with $J \stackrel{\rm def}{=} \det(\partial x^{\mu'}/\partial x^\nu)$, in which the transformed $\mathbf{\Psi}_{\mu'}{}^{\nu'\lambda'}$ is known from Eqs.(2·1), (2·36),(4·144),(4·145) and (4·146).

The relation

$$M^c{}_{\mu'} = 0 = (C^{-1})^\nu{}_\mu M^c{}_\nu \qquad (4\cdot161)$$

holds without any additional condition on the positive parameters p , s , β , γ and u, and $L_\mu{}^\nu$ and $L_{[\mu\nu]}$ transform according as

$$L_{\mu'}{}^{\nu'} = (C^{-1})^\rho{}_\mu C^\nu{}_\sigma L_\rho{}^\sigma \ , \qquad (4\cdot162)$$
$$L_{[\mu'\nu']} = 0 = (C^{-1})^\lambda{}_\mu (C^{-1})^\rho{}_\nu L_{[\lambda\rho]} \ , \qquad (4\cdot163)$$

under the condition

$$p > 1 \ . \qquad (4\cdot164)$$

Equations (4·161),(4·162) and (4·163) are the transformation rules which $M^c{}_\mu, L_\mu{}^\nu$ and $L_{[\mu\nu]}$ are desirable to obey.

4.1.4. *Charge*

For the transformed charge

$$C' \stackrel{\rm def}{=} \int_\sigma \partial_{\nu'}\left(\frac{\partial \mathbf{L}'}{\partial A_{\mu',\nu'}}\right) d\sigma_{\mu'} \qquad (4\cdot165)$$

with the Lagrangian density $\mathbf{L}'$ defined with transformed field variables, we can obtain

$$C' = C = q \ , \qquad (4\cdot166)$$

without imposing any additional condition.

To summarize, G.E.P. is satisfied, if

$$p > 1 \ , \quad s > 2 \ . \qquad (4\cdot167)$$

The asymptotic behaviors of the transformed dual components of $b^{k\prime}{}_{\mu'} = H^k{}_l(\partial x^\nu/\partial x^{\mu'})b^l{}_\nu$ and of transformed vector potential $A_{\mu'} = (\partial x^\nu/\partial x^{\mu'})A_\nu$ are easily known from Eqs.(4·126),(4·128), (4·142) and (4·167) as

$$b^{k\prime}{}_{\mu'} = \Lambda^k{}_l (C^{-1})^\nu{}_\mu {}^{(0)}b^l{}_\nu + O^k{}_\mu(1/r^u) + O^k{}_\mu(1/r) \ , \quad (u>0) \ ,$$
$$A_{\mu'} = (C^{-1})^\nu{}_\mu A_\nu + d^\nu{}_\mu A_\nu = O_\mu(1/r) \ . \qquad (4\cdot168)$$

4.2. *The case in which $\{\psi^k, b^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables*

4.2.1. *Energy-momentum and angular momentum*

For the generator of Poincaré gauge transformations, we have

$$\hat{M}_{k'} \stackrel{\rm def}{=} \int_\sigma {}^{\rm tot}\hat{\mathbf{T}}_{k'}{}^{\mu'} d\sigma_{\mu'} \equiv 0 = \Lambda_k{}^l \hat{M}_l \ , \qquad (4\cdot169)$$
$$\hat{S}_{k'l'} \stackrel{\rm def}{=} \int_\sigma {}^{\rm tot}\mathbf{S}_{k'l'}{}^{\mu'} d\sigma_{\mu'} = \Lambda_k{}^m \Lambda_l{}^n \hat{S}_{mn} \ , \qquad (4\cdot170)$$

which hold without imposing any additional condition.

4.2.2. *Canonical energy-momentum and "extended orbital angular momentum"*

For $\hat{M}^c{}_\mu$, we have

$$\hat{M}^c{}_{\mu'} \stackrel{\text{def}}{=} \int_\sigma \hat{\mathbf{T}}_{\mu'}{}^{\nu'} d\sigma_{\nu'} = (C^{-1})^\nu{}_\mu \hat{M}^c{}_\nu \ , \tag{4·171}$$

if the condition

$$p > \frac{1}{2} \ , \quad s > 1 \ , \quad p + u > 1 \tag{4·172}$$

is satisfied. The transformed extended orbital angular momentum

$$\hat{L}_{\mu'}{}^{\nu'} \stackrel{\text{def}}{=} \int_\sigma \hat{\mathbf{M}}_{\mu'}{}^{\nu'\lambda'} d\sigma_{\lambda'} \tag{4·173}$$

is divergent in general, as is obvious from Eq.(3·120). But, the transformed orbital angular momentum $\hat{L}_{[\mu'\nu']}$ and hence the transformed total angular momentum $\hat{J}_{k'l'} \stackrel{\text{def}}{=} \hat{S}_{k'l'} + {}^{(0)}b^{\mu'}{}_{k'}{}^{(0)}b^{\nu'}{}_{l'}\hat{L}_{[\mu'\nu']}$ are well defined, and they obey the rules

$$\hat{L}_{[\mu'\nu']} = (C^{-1})^\lambda{}_\mu (C^{-1})^\rho{}_\nu \hat{L}_{[\lambda\rho]} \ , \tag{4·174}$$

$$\hat{J}_{k'l'} = \Lambda_k{}^m \Lambda_l{}^n \hat{J}_{mn} \ , \tag{4·175}$$

if the condition

$$p > \frac{1}{2} \ , \quad s > 2 \ , \quad u > 1 \ , \quad p + u > 2 \tag{4·176}$$

is satisfied.

4.2.3. *Charge*

The transformed charge

$$\hat{C}' \stackrel{\text{def}}{=} \int_\sigma \partial_{\nu'} \left(\frac{\partial \hat{\mathbf{L}}'}{\partial A_{\mu',\nu'}} \right) d\sigma_{\mu'} \tag{4·177}$$

with the Lagrangian density $\hat{\mathbf{L}}'$ defined with transformed field variables is evaluated as

$$\hat{C}' = \hat{C} = q \ , \tag{4·178}$$

without imposing any additional condition.

To summarize, G.E.P. is satisfied, if the condition (4·176) is satisfied. The asymptotic behaviors of the transformed dual components of $\hat{b}^{k'}{}_{\mu'} \stackrel{\text{def}}{=} H^k{}_l(\partial x^\nu/\partial x^{\mu'})b^l{}_\nu$ and of the transformed vector potential $\hat{A}_{\mu'} \stackrel{\text{def}}{=} (\partial x^\nu/\partial x^{\mu'})A_\nu$ are easily known from Eqs.(4·126),(4·128), (4·142) and (4·176) as

$$\begin{aligned} \hat{b}^{k'}{}_{\mu'} &= \Lambda^k{}_l (C^{-1})^\nu{}_\mu {}^{(0)}b^l{}_\nu + O^k{}_\mu(1/r^p) \ , \quad (p > \frac{1}{2}) \ , \\ \hat{A}_{\mu'} &= (C^{-1})^\nu{}_\mu A_\nu + d^\nu{}_\mu A_\nu = O_\mu(1/r) \ . \end{aligned} \tag{4·179}$$

§5. Summary and discussions

In extended new general relativity (E.N.G.R.), we have examined exact charged axisymmetric solutions of the gravitational and electromagnetic field equations in vacuum from the point of view of equivalence principle.

In this theory, the generators depend on the choice of the set of independent field variables. In §3., we have examined the solution given by Eqs.(2·67) and (2·68), for the case in which $\{\psi^k, A^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables and for the case in which $\{\psi^k, b^k{}_\mu, A_\mu\}$ is employed as the set of independent variables. We have shown the following:

(A) For the case in which $\{\psi^k, A^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables: The total energy-momentum, the total angular momentum and the total electric charge of the system are all given by generators of *internal* transformations. The canonical energy-momentum and the orbital angular momentum are vanishing and trivial.

(B) For the case in which $\{\psi^k, b^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables: (1) The total energy-momentum is given by the generator $M^c{}_\mu$ of the coordinate translations, and the generator $\hat{M}_k$ of the internal translations vanishes identically. (2) The total angular momentum is given by the sum $\hat{J}_{kl} \stackrel{\text{def}}{=} \hat{S}_{kl} + {}^{(0)}b^\mu{}_k{}^{(0)}b^\nu{}_l \hat{L}_{[\mu\nu]}$ of the generator $\hat{S}_{kl}$ of the internal Lorentz transformations and of the generator $\hat{L}_{[\mu\nu]}$ of the coordinate Lorentz transformations. (3) The total charge is given by the generators of the internal $U(1)$ transformations.

(A∩B) For both cases, the total energy-momentum, the total angular momentum and the total charge of the system agree with the corresponding active quantities of the central gravitating body.

The total mass, which is equal to the total energy divided by the square of velocity of light, can be regarded as the inertial mass of the system. Thus, the results mentioned above include the equality of the inertial mass and the active gravitational mass which implies that an equivalence principle is satisfied by the solution given by Eqs.(2·67) and (2·68).

In view of this, we have introduced the notion of a generalized equivalence principle (G.E.P.) as stated in the beginning of §4. Solutions obtained by applying {the transformations (4·126) and (4·127) satisfying the conditions (4·125) and (4·139)} $\otimes$ { coordinate transformation (4·140)} to the original solution have been examined, from the point of view of G.E.P. The following results have been obtained.

(A′) For the case in which $\{\psi^k, A^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables, G.E.P. is satisfied by solutions, if the condition (4·167) is satisfied.

(B′) For the case in which $\{\psi^k, b^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables, G.E.P. is satisfied, if the condition (4·176) is satisfied.

We would like to add several comments:

[1] For the internal transformation (4·126), the condition (4·167) gives a stronger condition than the condition (4·176) does. For the coordinate transformation (4·140), no constraint is imposed by the former, while the latter gives the restriction $u > 1, p + u > 2$.

[2] For the case in which $\{\psi^k, A^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables, G.E.P. *is satisfied even by solutions which approach constant values very slowly at spatial infinity*, as is seen from Eq.(4·168).
This is a direct consequence of the following:

(a) *The functions* $\mathbf{F_k}^{\mu\nu}$ *and* $\mathbf{F}^{\mu\nu} = \sqrt{-g}F^{\mu\nu}$ *both describe tensor densities*, and hence the total energy-momentum M_k, total angular momentum S_{kl} and the total charge C are all independent of coordinate systems employed.

(b) The canonical energy-momentum $M^c{}_\mu$ and the extended orbital angular momentum $L_\mu{}^\nu$, which are the generators of coordinate transformations, obey the regular transformation rules (4·161) and (4·162) under the coordinate transformation (4·140) with *arbitrary positive* u, if the condition (4·164) is satisfied.

Note that $b^{k\prime}{}_{\mu\prime}$ as given by Eq.(4·168) approaches constant value much slower than the vierbein components required in general situation to give reasonable energy-momentum and angular momentum.[6)]

[3] In the case in which $\{\psi^k, b^k{}_\mu, A_\mu\}$ is employed as the set of independent field variables, G.E.P. for the total energy-momentum and for the total angular momentum is established when the vierbeins approach constants with the order $O(1/r^p)$, $(p > 1/2)$ at spatial infinity. This is consistent with the results[5)] on the equivalence principle for the energy in new general relativity (N.G.R.). [*)]

The preceding results, together with those in Ref. 6, show that the choice $\{\psi^k, A^k{}_\mu, A_\mu\}$ as the set of independent field variables is preferential to the choice $\{\psi^k, b^k{}_\mu, A_\mu\}$. This is quite natural, because the fields $\psi^k, A^k{}_\mu$ and A_μ are the fundamental objects and $b^k{}_\mu$ is a composite of ψ^k and of $A^k{}_\mu$.

Appendix A

—— *Asymptotic forms of* $\mathbf{F}^{(1)}{}_k{}^{\mu\nu}$, $V^{mn}{}_\lambda$ *and* $W^{\mu\nu\lambda}{}_{klm}$ *for large* r ——

We give the asymptotic forms of $\mathbf{F}^{(1)}{}_k{}^{\mu\nu}$, $V^{mn}{}_\lambda$ and $W^{\mu\nu\lambda}{}_{klm}$ for large r which are useful for calculations in §3. and in §4.

$\mathbf{F}^{(1)}{}_k{}^{\mu\nu}$:

$$\mathbf{F}^{(1)}{}_{(0)}{}^{0\alpha} = -\frac{1}{\kappa}\left(a - \frac{Q^2}{r}\right)\frac{n^\alpha}{r^2} + \frac{ah}{2\kappa r^3}\varepsilon^\alpha{}_{\beta 3}n^\beta + O^\alpha\left(\frac{1}{r^4}\right) ,$$

$$\begin{aligned}\mathbf{F}^{(1)}{}_a{}^{0\alpha} &= -\frac{1}{\kappa r^2}\left(a - \frac{3Q^2}{2r}\right)n_a n^\alpha - \frac{Q^2}{2\kappa r^3}\delta_a{}^\alpha \\ &\quad -\frac{ah}{2\kappa r^3}\left(\delta^{\alpha\beta} - 3n^\alpha n^\beta\right)\varepsilon_{a\beta 3} + O_a{}^\alpha\left(\frac{1}{r^4}\right) ,\end{aligned}$$

*) Note that E.N.G.R. is reduced to N.G.R., if fields with nonvanishing P_k do not take part in and if the set $\{\psi^k, A^k{}_\mu, A_\mu, \phi^A, \phi^{*A}\}$ is employed as the set of independent field variables.

$$
\begin{aligned}
\mathbf{F}^{(1)}{}_{(0)}{}^{\alpha\beta} &= \frac{3ah}{2\kappa r^3}\left(n^\alpha \varepsilon^\beta{}_{\gamma 3} - n^\beta \varepsilon^\alpha{}_{\gamma 3}\right) n^\gamma + \frac{ah}{\kappa r^3}\varepsilon^{\alpha\beta}{}_3 + O^{[\alpha\beta]}\left(\frac{1}{r^4}\right) , \\
\mathbf{F}^{(1)}{}_a{}^{\alpha\beta} &= -\frac{ah}{2\kappa r^3}\left(\varepsilon_a{}^\alpha{}_3 n^\beta - \varepsilon_a{}^\beta{}_3 n^\alpha\right) + \frac{2ah}{\kappa r^3}\left(n^\alpha \varepsilon^\beta{}_{\gamma 3} - n^\beta \varepsilon^\alpha{}_{\gamma 3}\right) n_a n^\gamma \\
&\quad + \frac{ah}{\kappa r^3}\varepsilon^{\alpha\beta}{}_3 n_a - \frac{Q^2}{2\kappa r^3}\left(\delta_a{}^\alpha n^\beta - \delta_a{}^\beta n^\alpha\right) + O_a{}^{[\alpha\beta]}\left(\frac{1}{r^4}\right) . \qquad (\mathrm{A}\cdot 1)
\end{aligned}
$$

$V^{mn}{}_\lambda$:

$$
\begin{aligned}
V^{(0)a}{}_0 &= -n^a \dot{\Xi} - a\frac{n^a}{r}\dot{\Pi} - \frac{n^a}{2}\Xi^2(\dot{\Xi} - \frac{a}{r}\dot{\Pi}) + a\frac{n^a}{r}\Xi\Pi\dot{\Xi} \\
&\quad + O^a\left(\frac{1}{r^{s+1}}\right) + O^a\left(\frac{1}{r^{2p+2}}\right) + o^a\left(\frac{1}{r^3}\right) , \\
V^{(0)a}{}_\alpha &= -\frac{1}{r}\left(\delta^a{}_\alpha - n^a n_\alpha\right)\Xi - n^a n_\alpha \Xi' - \frac{a}{r^2}\left(\delta^a{}_\alpha - 2n^a n_\alpha\right)\Pi \\
&\quad - \frac{a n^a n_\alpha}{r}\Pi' + \frac{n^a n_\alpha}{2}\Xi^2\Xi' + O^a{}_\alpha\left(\frac{1}{r^{s+1}}\right) \\
&\quad + O^a{}_\alpha\left(\frac{1}{r^{2p+2}}\right) + o^a{}_\alpha\left(\frac{1}{r^3}\right) , \\
V^{ab}{}_0 &= -\frac{a}{r}n^a n^b(\Xi\dot{\Pi} + \dot{\Xi}\Pi) - a^2\frac{n^a n^b}{r^2}\Pi\dot{\Pi} + \frac{1}{2}n^a n^b \Xi^3\dot{\Xi} \\
&\quad + O^{ab}\left(\frac{1}{r^{s+1}}\right) + O^{ab}\left(\frac{1}{r^{2p+2}}\right) + o^{ab}\left(\frac{1}{r^3}\right) , \\
V^{ab}{}_\alpha &= -\frac{1}{r}(n^a \delta^b{}_\alpha - n^b \delta^a{}_\alpha)\left(1 - \frac{1}{4}\Xi^2\right)\Xi^2 + O^{ab}{}_\alpha\left(\frac{1}{r^{s+1}}\right) \\
&\quad + O^{ab}{}_\alpha\left(\frac{1}{r^{2p+2}}\right) + o^{ab}{}_\alpha\left(\frac{1}{r^3}\right) \qquad (\mathrm{A}\cdot 2)
\end{aligned}
$$

with [*)]

$$
\Xi \stackrel{\mathrm{def}}{=} A' + \dot{B} , \quad \Pi \stackrel{\mathrm{def}}{=} \dot{A} + \dot{B} - A' - B' + \frac{1}{r}(A+B) , \qquad (\mathrm{A}\cdot 3)
$$

and $o^a{}_\alpha(1/r^3)$, for example, denotes a term such that $\lim_{r\to\infty} r^3 o^a{}_\alpha(1/r^3) = 0$.

$W^{\mu\nu\lambda}{}_{klm}$:

$$
\begin{aligned}
W^{0\alpha\beta}{}_{(0)ab} &= \frac{1}{2}\left(\delta^\alpha{}_a\delta^\beta{}_b - \delta^\alpha{}_b\delta^\beta{}_a\right) + \frac{a}{4r}\left\{\left(\delta^\alpha{}_a\delta^\beta{}_b - \delta^\alpha{}_b\delta^\beta{}_a\right) + \left(n_a\delta^\alpha{}_b - n_b\delta^\alpha{}_a\right)n^\beta \right. \\
&\quad \left. + \left(n_b\delta^\beta{}_a - n_a\delta^\beta{}_b\right)n^\alpha\right\} + O^{[\alpha\beta]}{}_{[ab]}\left(\frac{1}{r^2}\right) , \\
W^{0\alpha\beta}abc &= \frac{a}{4r}\left\{\left(\delta^\alpha{}_a\delta^\beta{}_b - \delta^\alpha{}_b\delta^\beta{}_a\right)n_c + \left(\delta^\alpha{}_b\delta^\beta{}_c - \delta^\alpha{}_c\delta^\beta{}_b\right)n_a + \left(\delta^\alpha{}_c\delta^\beta{}_a - \delta^\alpha{}_a\delta^\beta{}_c\right)n_b\right\} \\
&\quad + O^{[\alpha\beta]}{}_{[abc]}\left(\frac{1}{r^2}\right) ,
\end{aligned}
$$

[*)] Here, the symbols $'$ and $\dot{}$ mean the derivatives with respect to r and x^0, respectively. For example, $A' \stackrel{\mathrm{def}}{=} \partial A/\partial r$, $\dot{A} \stackrel{\mathrm{def}}{=} \partial A/\partial x^0$.

$$
\begin{aligned}
W^{\alpha\beta\gamma}{}_{(0)ab} = & -\frac{a}{4r}\left\{\left(\delta^{\alpha}{}_{a}\delta^{\beta}{}_{b}-\delta^{\alpha}{}_{b}\delta^{\beta}{}_{a}\right)n^{\gamma}+\left(\delta^{\beta}{}_{a}\delta^{\gamma}{}_{b}-\delta^{\beta}{}_{b}\delta^{\gamma}{}_{a}\right)n^{\alpha}+\left(\delta^{\gamma}{}_{a}\delta^{\alpha}{}_{b}-\delta^{\gamma}{}_{b}\delta^{\alpha}{}_{a}\right)n^{\beta}\right\} \\
& +O^{[\alpha\beta\gamma]}{}_{[ab]}\left(\frac{1}{r^{2}}\right) , \\
W^{\alpha\beta\gamma}{}_{abc} = & \frac{1}{2}\left\{\left(\delta^{\alpha}{}_{a}\delta^{\beta}{}_{b}-\delta^{\beta}{}_{a}\delta^{\alpha}{}_{b}\right)\delta^{\gamma}{}_{c}+\left(\delta^{\alpha}{}_{b}\delta^{\beta}{}_{c}-\delta^{\beta}{}_{b}\delta^{\alpha}{}_{c}\right)\delta^{\gamma}{}_{a}+\left(\delta^{\alpha}{}_{c}\delta^{\beta}{}_{a}-\delta^{\beta}{}_{c}\delta^{\alpha}{}_{a}\right)\delta^{\gamma}{}_{b}\right\} \\
& -\frac{a}{4r}\left\{\left(\delta^{\alpha}{}_{a}\delta^{\beta}{}_{b}-\delta^{\beta}{}_{a}\delta^{\alpha}{}_{b}\right)n_{c}+\left(\delta^{\alpha}{}_{b}\delta^{\beta}{}_{c}-\delta^{\beta}{}_{b}\delta^{\alpha}{}_{c}\right)n_{a}+\left(\delta^{\alpha}{}_{c}\delta^{\beta}{}_{a}-\delta^{\beta}{}_{c}\delta^{\alpha}{}_{a}\right)n_{b}\right\}n^{\gamma} \\
& -\frac{a}{4r}\left[\left\{\left(\delta^{\alpha}{}_{a}n_{b}-\delta^{\alpha}{}_{b}n_{a}\right)n^{\beta}-\left(\delta^{\beta}{}_{a}n_{b}-\delta^{\beta}{}_{b}n_{a}\right)n^{\alpha}\right\}\delta^{\gamma}{}_{c}\right. \\
& +\left\{\left(\delta^{\alpha}{}_{b}n_{c}-\delta^{\alpha}{}_{c}n_{b}\right)n^{\beta}-\left(\delta^{\beta}{}_{b}n_{c}-\delta^{\beta}{}_{c}n_{b}\right)n^{\alpha}\right\}\delta^{\gamma}{}_{a} \\
& \left.+\left\{\left(\delta^{\alpha}{}_{c}n_{a}-\delta^{\alpha}{}_{a}n_{c}\right)n^{\beta}-\left(\delta^{\beta}{}_{c}n_{a}-\delta^{\beta}{}_{a}n_{c}\right)n^{\alpha}\right\}\delta^{\gamma}{}_{b}\right]+O^{[\alpha\beta\gamma]}{}_{[abc]}\left(\frac{1}{r^{2}}\right) . \qquad (\mathrm{A}\cdot 4)
\end{aligned}
$$

All the other components, which are not listed in the above, are equal to zero or are obtainable from the listed ones by permutations.

References

1) C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p.448.
2) T. Shirafuji, G. G. L. Nashed and Y. Kobayashi, *Prog.Theor.Phys.* **96** (1996), 933.
3) Y. Bozhkov and W. A. Rodrigues, Jr., *Gen. Rel. Gravit.* **27** (1995), 813.
4) K. Hayashi and T. Shirafuji, *Phys.Rev.* **D19** (1979), 3524.
5) T. Shirafuji, G. G. L. Nashed and K. Hayashi, *Prog.Theor.Phys.* **95** (1996), 665.
6) T. Kawai and N. Toma, *Prog.Theor.Phys.* **85** (1991), 901.
7) T. Kawai, *Gen. Rel. Gravit.* **18** (1986), 995; **19** (1987), 1285 E.
8) T. Kawai, *Prog.Theor.Phys.* **82** (1989), 850.
9) T. Kawai and H. Saitoh, *Prog.Theor.Phys.* **81** (1989), 280;1119.
10) T. Kawai and N. Toma, *Prog.Theor.Phys.* **87** (1992), 583.
11) R. Utiyama, *Phys.Rev.* **101** (1956), 1597.
12) T. Kawai, *Prog.Theor.Phys.* **79** (1988), 920.